Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 24, 2003*
Revised: *June 27, 2003*
Accepted: *July 4, 2003*

The problem of large leptonic mixing

Joaquim I. Silva-Marcos*

Theoretical Physics Division, CERN
CH-1211 Geneva 23, Switzerland
E-mail: Joaquim.Silva-Marcos@cern.ch

Abstract: Unlike in the quark sector where simple S_3 permutation symmetries can generate the general features of quark masses and mixings, we find it impossible (under conditions of hierarchy for the charged leptons and without considering the see-saw mechanism or a more elaborate extension of the SM) to guarantee large leptonic mixing angles with any general symmetry or transformation of only known particles. If such symmetries exist, they must be realized in more extended scenarios.

Keywords: Beyond Standard Model, Neutrino Physics.

*On leave from Instituto Superior Técnico, Centro de Física das Interacções Fundamentais (CFIF), Lisbon, Portugal.

Recent neutrino data [1] have not only provided clear evidence pointing towards neutrino oscillations with very large mixing angles and non-zero neutrino masses, but have also added more questions to one of the most intriguing puzzles in particle physics: the flavour problem. We do not know of any fundamental theory of flavour, but several specific patterns for the fermion mass matrices have been proposed that account for the data. Our hope is to find some pattern that may point towards the existence of some family symmetry at a higher energy scale [2]. For instance, in the quark sector, an $S^q_{3\mathrm{L}} \times S^u_{3\mathrm{R}} \times S^d_{3\mathrm{R}}$ family permutation symmetry (acting on the left-handed quark doublets, the right-handed up quarks and the right-handed down quarks) automatically leads to quark mass matrices M_u, M_d proportional to the so-called democratic mass matrix [3], which has all elements equal to unity. In the democratic limit, only the third generation acquires mass and the Cabibbo-Kobayashi-Maskawa (CKM) matrix is the unit matrix. This is a remarkable result. Experimentally, one knows that there is a strong hierarchy in the value of the quark masses. The first two generations of quarks are much lighter than the third one, and the observed CKM matrix is close to the unit matrix. When the permutation symmetry is broken, the first two generations acquire non-vanishing masses, and a non-trivial CKM matrix is generated.

Inspired by this, one may be tempted to try and find some symmetry for the lepton sector. However, one is then confronted with the problem of generating the large leptonic mixing. Let us assume, e.g. as in [4], that the large mass difference between the neutrinos and the other leptons comes from a Majorana Yukawa term in the Lagrangian:

$$-\mathcal{L} = \frac{\lambda^\nu_{ij}}{M}\, \phi^\dagger \phi^\dagger L_i L_j + \lambda^e_{ij}\, \overline{L}_i \phi e_{jR} + \text{h.c.}\,, \tag{1}$$

where the L_i are the left-handed doublets, ϕ the Higgs field, e_{jR} the right-handed charged lepton singlets and M a large mass. As in the case for the quarks, an $S_{3\mathrm{L}} \times S_{3\mathrm{R}}$ symmetry, acting on the left-handed lepton doublets and the right-handed charged lepton singlets, leads to a charged lepton mass matrix proportional to the democratic matrix,[1] denoted by Δ. However, the most general mass matrix obtained from the Majorana term in eq. (1) form $\lambda\Delta + \mu\mathbb{1}$. As was pointed out [4], there is no reason to expect that λ and μ should not be of the same order of magnitude. As a result, both the charged lepton mass matrix and the neutrino mass matrix are, in leading order, diagonalized by the same unitary matrix, and the leptonic mixing matrix will just be the unit matrix. It is clear, unless one puts $\lambda = 0$ by hand, that no large angles can be generated by a small breaking of the $S_{3\mathrm{L}} \times S_{3\mathrm{R}}$ symmetry. By making the ad-hoc assumption that the coefficient λ vanishes, one can, of course, obtain the required large lepton mixing [5], but this is not dictated by the symmetry. More precisely, the Lagrangian does not acquire any new symmetry in the limit where λ vanishes. Therefore, setting $\lambda = 0$ clearly violates 't Hooft's naturalness principle [7].

In this Letter we shall prove that this problem for the leptons is, indeed, much more general. In particular, one could imagine that some other symmetry (or representation of

[1]It must be enphasized that there exist many other symmetries (which are not permutations and) that require the charged lepton mass matrix to be proportional to the democratic matrix, $M_e = \lambda\Delta$, e.g. a $\mathbb{Z}_3$ symmetry as proposed in ref. [9].

the leptons) might exist that would require the charged lepton mass matrix to be proportional to Δ, while at the same time preventing the neutrino Majorana mass from acquiring a similar term. We shall prove that this is impossible. Unlike in the quark sector, where small mixing results from a (permutation) symmetry, in the lepton sector, large mixing does not explicitly follow from any symmetry.

Let us assume that the left-handed doublets and right-handed charged lepton singlets transform under some general symmetry as

$$\begin{aligned} L_i \quad &\rightarrow \quad P_{ij}\ L_j \\ e_{iR} \quad &\rightarrow \quad Q_{ij}\ e_{jR} \,. \end{aligned} \tag{2}$$

The charged lepton and neutrino mass matrices must then be invariant under

$$P^\dagger \cdot M_e \cdot Q = M_e\,; \qquad P^T \cdot M_\nu \cdot P = M_\nu\,. \tag{3}$$

We assume from this relation that M_e is some general rank 1 matrix. This is to be expected because of the hierarchy of the charged leptons. Two cases are possible:

1. There is no correlation between the symmetry and the mixing; M_e is not completely determined, but the sum of (more then one unique) rank 1 matrices. This is the case, e.g. if $P = \mathbb{1}$, $Q = \mathrm{diag}(-1,-1,1)$, or $P = \mathrm{diag}(i,1,1)$, $Q = \mathrm{diag}(-1,-1,1)$. For these special cases, the mixing can be large but is arbitrary. As an example, take $P = \mathrm{diag}(i,1,1)$, $Q = \mathrm{diag}(-1,-1,1)$, then

$$M_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b_e \\ 0 & 0 & c_e \end{bmatrix}; \qquad M_\nu = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_\nu & b_\nu \\ 0 & b_\nu & c_\nu \end{bmatrix}. \tag{4}$$

   It is clear that the mixing dependends (also) on the arbitrary ratio b_e/c_e. Thus, e.g. maximal (atmospheric) mixing is not explicitly guarranteed by the symmetry but has to be put by hand, by fine-tunning several (unrelated) parameters from the charged lepton and neutrino sector.

2. The mixing is small. This is the case when eq. (3) implies that M_e is proportional to just one unique rank 1 matrix,[2] $M_e \sim \Delta$. Then, in particular, one must have $P^\dagger \cdot \Delta \cdot Q = \Delta$. Squaring and using $\Delta^2 = 3\Delta$, one finds $P^\dagger \cdot \Delta \cdot P = \Delta$, which means that $P \cdot \Delta = \Delta \cdot P$ and thus $P^T \cdot \Delta = \Delta \cdot P^T$. Using again eq. (3), one can now also find extra relations for M_ν:

$$\begin{aligned} P^T \cdot \Delta M_\nu \cdot P &= \Delta \cdot P^T \cdot M_\nu \cdot P = \Delta M_\nu \\ P^T \cdot M_\nu \Delta \cdot P &= P^T \cdot M_\nu \cdot P \cdot \Delta = M_\nu \Delta \\ P^T \cdot \Delta M_\nu \Delta \cdot P &= \Delta \cdot P^T \cdot M_\nu \cdot P \cdot \Delta = \Delta M_\nu \Delta\,. \end{aligned} \tag{5}$$

[2] We shall work here in a democratic weak basis. Obviously, our statement can be extended to any other convenient basis.

Therefore, whatever the neutrino mass matrix is, nothing prevents it from having additional large parts which can be written as:

$$M_\nu + \lambda' \ (\Delta M_\nu + M_\nu \Delta) + \lambda \ \Delta M_\nu \Delta \,. \tag{6}$$

Note that, for any matrix X, one has $\Delta X \Delta = x \ \Delta$, where $x = \sum X_{ij}$. Thus, M_ν can have a large part proportional to Δ. As a consequence, there exists no symmetry (be it discrete or not, but realized in the form given in eq. (2)), that, as in ref. [5], would force the neutrino mass matrix to be strictly proportional to $\mathbb{1}$. It will also have a part proportional to Δ. In general, writing the neutrino mass matrix as $M_\nu = A + \lambda\Delta$, where A and λ are of the same order, the symmetry cannot guarantee the existence of large mixing angles necessary to solve the atmospheric neutrino problem, because the term with Δ will be under no restriction from the symmetry.

In fact, we can even be more precise and find very severe constraints for the neutrino mass matrix and the mixing angles. From eq. (3) for the neutrino mass matrix one concludes that M_ν must also satisfy, $P^\dagger \cdot M_\nu^\dagger M_\nu \cdot P = M_\nu^\dagger M_\nu$. Combining this with $P^\dagger \cdot \Delta \cdot Q = \Delta$, we find the relation:

$$P^\dagger \cdot M_\nu^\dagger M_\nu \Delta \cdot Q = M_\nu^\dagger M_\nu \Delta \,. \tag{7}$$

This seems to be an equation for the neutrino mass matrix, but, actually, one must realize that it obeys the set of conditions that define the mass matrix of the charged leptons as formulated in eq. (3). Therefore, $M_\nu^\dagger M_\nu \Delta$ must be proportional to the charged lepton mass matrix i.e. to Δ:

$$M_\nu^\dagger M_\nu \Delta = p\Delta \,. \tag{8}$$

Finally, we conclude that, because of the symmetry, the mass matrices of the neutrinos and the charged leptons are intrinsically related. It is clear that this has very strong consequences for the lepton mixing. It follows that the matrix F that diagonalizes Δ (i.e. the charged leptons on the right) must also partially diagonalize $M_\nu^\dagger M_\nu$. Using eq. (8), one finds that the matrix that diagonalizes $M_\nu^\dagger M_\nu$ can be written as $F \cdot U$, where U is a simple unitary matrix with only significant elements in the 2×2 sector, i.e. $U_{13} = U_{23} = 0$. As a result, the lepton mixing matrix will be just this U. No perturbation, breaking the symmetry and giving small contributions to M_ν and M_e, would be sufficient to obtain the large mixing angles needed to solve the atmospheric neutrino problem.[3]

One may try to avoid this difficulty and find less severe conditions, e.g. by requiring that the charged leptons mass matrix be, not strictly proportional to Δ but, only hierarchical: $M_e = V \Delta W$, where V and W are unitary matrices. But then, using analogous arguments,

[3]There could be one exception to this result: if the symmetry could force M_ν to be exactly proportional to a unitary matrix. Then, the diagonalization of M_ν would not coincide with the diagonalization of $M_\nu^\dagger M_\nu$, e.g. $M_\nu = (\omega - 1)\mathbb{1} + \Delta$, where $\omega = e^{2\pi i/3}$. It would be possible to get large mixing angles, because, in this specific case, small perturbations of the neutrino mass matrix would cause singular effects [6]. However, one must realize that unitarity can only be obtained with a very special combination of terms, like setting $\lambda = 0$ in our previous $S_{3\mathrm{L}} \times S_{3\mathrm{R}}$ symmetry example. As we have argued here, unitarity can never be forced by the symmetry, because it is always possible to add any (small or large) term proportional to Δ to M_ν and this would just lead to our small mixing angles result.

we find, instead of eq. (8), the relation $M_\nu^\dagger M_\nu V\Delta V^\dagger = pV\Delta V^\dagger$ between the neutrino mass matrix and the charged lepton square mass matrix $H_e = M_e M_e^\dagger = 3\ V\Delta V^\dagger$. Clearly, this will just lead to the same result. It is equivalent to an irrelevant change of weak basis. Another possibility would be to state that neutrinos are of the Dirac type.[4] The relation for the neutrino mass matrix in eq. (3) will then read $P^\dagger \cdot M_\nu \cdot R = M_\nu$, where R is some transformation of the right-handed neutrinos. We will then find that $M_\nu M_\nu^\dagger \Delta = p\Delta$. Again, we are unable to guarantee large mixing angles from the symmetry or by a small breaking of it.

We find it, therefore, impossible (under the conditions stated with regard to the hierarchy and mass matrix of the charged leptons and without considering the see-saw mechanism) to guarantee large mixing angles in the leptonic sector with any general symmetry or transformation of only known leptons. If such symmetries exist, they must be realized in more extended scenarios [8]. One may also interpret this result as a requirement to go beyond the SM and it is indeed possible to construct a $\mathbb{Z}_3$ permutation symmetry that, in the context of the see-saw model, can lead to large mixing angles [9].

Acknowledgments

I am grateful to G.C. Branco for a thorough reading of the manuscript and suggestions. This work received partial support from the Portuguese Ministry of Science — Fundação para a Ciência e Tecnologia and the CERN Theory Division which I thank for the kind hospitality extended.

References

[1] MACRO collaboration, M. Ambrosio et al., *Matter effects in upward-going muons and sterile neutrino oscillations*, *Phys. Lett.* **B 517** (2001) 59 [hep-ex/0106049];
SUPER-KAMIOKANDE collaboration, S. Fukuda et al., *Solar* ^{8}B *and hep neutrino measurements from 1258 days of super-Kamiokande data*, *Phys. Rev. Lett.* **86** (2001) 5651 [hep-ex/0103032];
SUPER-KAMIOKANDE collaboration, S. Fukuda et al., *Constraints on neutrino oscillations using 1258 days of super-Kamiokande solar neutrino data*, *Phys. Rev. Lett.* **86** (2001) 5656 [hep-ex/0103033];
MACRO collaboration, M. Ambrosio et al., *Matter effects in upward-going muons and sterile neutrino oscillations*, *Phys. Lett.* **B 517** (2001) 59 [hep-ex/0106049];
GALLEX collaboration, W. Hampel et al., *Gallex solar neutrino observations: results for Gallex IV*, *Phys. Lett.* **B 447** (1999) 127;
SAGE collaboration, J.N. Abdurashitov et al., *Measurement of the solar neutrino capture rate with gallium metal*, *Phys. Rev.* **D 60** (1999) 055801 [astro-ph/9907113];
GNO collaboration, M. Altmann et al., *Gno solar neutrino observations: results for GNO I*, *Phys. Lett.* **B 490** (2000) 16 [hep-ex/0006034];
SNO collaboration, *Measurement of the rate of* $\nu e + d \to p + p + e$ *interactions produced by* ^{8}B *solar neutrinos at the Sudbury Neutrino Observatory*, *Phys. Rev. Lett.* **87** (2001) 71301.

[4] For simplicity, we do not consider (heavy) mass terms for the right-handed neutrinos and the see-saw mechanism. In fact, in the latter context, the statement we make here is not necessarily true [9].

[2] F. Vissani, *A study of the scenario with nearly degenerate Majorana neutrinos*, hep-ph/9708483;
E. Ma, *Dynamical neutrino mass matrix: large (small) mixing solution to atmospheric (solar) neutrino oscillations*, *Phys. Lett.* **B 442** (1998) 238 [hep-ph/9807386];
V.D. Barger, S. Pakvasa, T.J. Weiler and K. Whisnant, *Bi-maximal mixing of three neutrinos*, *Phys. Lett.* **B 437** (1998) 107 [hep-ph/9806387];
G. Altarelli and F. Feruglio, *Neutrino masses and mixings: a theoretical perspective*, *Phys. Rept.* **320** (1999) 295;
Z. Berezhiani and A. Rossi, *Predictive grand unified textures for quark and neutrino masses and mixings*, *Nucl. Phys.* **B 594** (2001) 113 [hep-ph/0003084];
J.A. Casas, V. Di Clemente, A. Ibarra and M. Quiros, *Massive neutrinos and the Higgs mass window*, *Phys. Rev.* **D 62** (2000) 053005 [hep-ph/9904295];
J.I. Silva-Marcos, *Alternative to the seesaw mechanism*, *Phys. Rev.* **D 59** (1999) 091301;
M. Jezabek and Y. Sumino, *Neutrino masses and bimaximal mixing*, *Phys. Lett.* **B 457** (1999) 139 [hep-ph/9904382];
R. Barbieri, G.G. Ross and A. Strumia, *Vacuum oscillations of quasi degenerate solar neutrinos*, *J. High Energy Phys.* **10** (1999) 020 [hep-ph/9906470];
R.N. Mohapatra and S. Nussinov, *Bimaximal neutrino mixing and neutrino mass matrix*, *Phys. Rev.* **D 60** (1999) 013002 [hep-ph/9809415];
S. Lola and G.G. Ross, *Neutrino masses from* U(1) *symmetries and the super- kamiokande data*, *Nucl. Phys.* **B 553** (1999) 81 [hep-ph/9902283];
B. Stech, *A predictive minimal model for neutrino masses and mixings*, *Phys. Rev.* **D 62** (2000) 093019 [hep-ph/0006076];
D. Black, A.H. Fariborz, S. Nasri and J. Schechter, *Complementary ansatz for the neutrino mass matrix*, *Phys. Rev.* **D 62** (2000) 073015 [hep-ph/0004105];
D. Falcone, *Heavy neutrino mass scale and quark-lepton symmetry*, *Phys. Lett.* **B 475** (2000) 92 [hep-ph/9912491];
G. Altarelli, F. Feruglio and I. Masina, *Large neutrino mixing from small quark and lepton mixings*, *Phys. Lett.* **B 472** (2000) 382 [hep-ph/9907532];
G. Altarelli, *Models of neutrino masses and mixings*, hep-ph/0106085; *Neutrino masses and grand unification*, *Nucl. Phys.* **87** *(Proc. Suppl.)* (2000) 291 [hep-ph/0001024];
H. Fritzsch and Z.-z. Xing, *Maximal neutrino mixing and maximal CP-violation*, *Phys. Rev.* **D 61** (2000) 073016 [hep-ph/9909304];
K. Fukuura, T. Miura, E. Takasugi and M. Yoshimura, *Maximal CP-violation, large mixings of neutrinos and democratic-type neutrino mass matrix*, *Phys. Rev.* **D 61** (2000) 073002 [hep-ph/9909415];
P.Q. Hung, *A model of quark and lepton masses. i: the neutrino sector*, *Phys. Rev.* **D 62** (2000) 053015 [hep-ph/0003303];
R. Adhikari, E. Ma and G. Rajasekaran, *Three inequivalent mass-degenerate Majorana neutrinos and a model of their splitting for neutrino oscillations*, *Phys. Lett.* **B 486** (2000) 134 [hep-ph/0004197];
S.M. Barr and I. Dorsner, *A general classification of three neutrino models and U_{e3}. Nucl. Phys.* **B 585** (2000) 79 [hep-ph/0003058];
A. Kageyama, S. Kaneko, N. Shimoyama and M. Tanimoto, *Lepton flavor violating processes in bi-maximal texture of neutrino mixings*, *Phys. Rev.* **D 65** (2002) 096010 [hep-ph/0112359];
E.K. Akhmedov, G.C. Branco, F.R. Joaquim and J.I. Silva-Marcos, *Neutrino masses and mixing with seesaw mechanism and universal breaking of extended democracy*, *Phys. Lett.* **B 498** (2001) 237 [hep-ph/0008010];
H.J. Pan and G. Cheng, *Seesaw mechanism in three flavors*, *Phys. Rev.* **D 65** (2002) 013002

[hep-ph/0103147];
N. Haba, J. Sato, M. Tanimoto and K. Yoshioka, *Phenomenology of neutrino mass matrix*, hep-ph/0102060;
K.R.S. Balaji, R.N. Mohapatra, M.K. Parida and E.A. Paschos, *Large neutrino mixing from renormalization group evolution*, *Phys. Rev.* **D 63** (2001) 113002 [hep-ph/0011263];
N. Haba, J. Sato, M. Tanimoto and K. Yoshioka, *Possible mixing patterns of charged leptons and neutrinos*, *Phys. Rev.* **D 64** (2001) 113016 [hep-ph/0101334];
R. Kitano and Y. Mimura, *Large angle msw solution in grand unified theories with* $SU(3) \times U(1)$ *horizontal symmetry*, *Phys. Rev.* **D 63** (2001) 016008 [hep-ph/0008269];
T. Miura, E. Takasugi and M. Yoshimura, *Large CP-violation, large mixings of neutrinos and the* Z_3 *symmetry*, *Phys. Rev.* **D 63** (2001) 013001 [hep-ph/0003139];
F. Feruglio, A. Strumia and F. Vissani, *Neutrino oscillations and signals in* β *and* $0\nu2\beta$ *experiments*, *Nucl. Phys.* **B 637** (2002) 345 [hep-ph/0201291];
M. Frigerio and A.Y. Smirnov, *Structure of neutrino mass matrix and CP-violation*, *Nucl. Phys.* **B 640** (2002) 233 [hep-ph/0202247];
S. Lavignac, I. Masina and C.A. Savoy, *Large solar angle and seesaw mechanism: a bottom-up perspective*, *Nucl. Phys.* **B 633** (2002) 139 [hep-ph/0202086].

[3] H. Fritzsch, *Composite quarks and leptons and their flavour mixing*, in *Proc. of Europhys. Conf. on Flavour Mixing in Weak Interactions* 1984, Erice, Italy;
G.C. Branco, J.I. Silva-Marcos and M.N. Rebelo, *Universal strength for yukawa couplings*, *Phys. Lett.* **B 237** (1990) 446;
G.C. Branco and J.I. Silva-Marcos, *Predicting V (CKM) with universal strength of Yukawa couplings*, *Phys. Lett.* **B 359** (1995) 166 [hep-ph/9507299];
P.M. Fishbane and P. Kaus, *Pure phase mass matrices*, *Phys. Rev.* **D 49** (1994) 3612; *Textured pure phase mass matrices*, *Z. Physik* **C 75** (1997) 1; *Nearly democratic mass matrices and flavour mixing*, *Phys. Rev.* **D 49** (1994) 4780;
G.C. Branco, D. Emmanuel-Costa and J.I. Silva-Marcos, *A new approach to the Yukawa puzzle*, *Phys. Rev.* **D 56** (1997) 107 [hep-ph/9608477];
P.M. Fishbane and P.Q. Hung, *Dynamics and symmetries for pure phase mass matrices*, *Phys. Rev.* **D 57** (1998) 2743;
J.I. Silva-Marcos, *Special symmetric quark mass matrices*, *Phys. Lett.* **B 443** (1998) 276 [hep-ph/9807549]; *Breaking democracy with non renormalizable mass terms*, hep-ph/0102079;
P.Q. Hung and M. Seco, *Pure phase mass matrices from six dimensions*, *Nucl. Phys.* **B 653** (2003) 123 [hep-ph/0111013].

[4] M. Fukugita, M. Tanimoto and T. Yanagida, *Atmospheric neutrino oscillation and a phenomenological lepton mass matrix*, *Phys. Rev.* **D 57** (1998) 4429 [hep-ph/9709388].

[5] H. Fritzsch and Z.-z. Xing, *Large leptonic flavor mixing and the mass spectrum of leptons*, *Phys. Lett.* **B 440** (1998) 313 [hep-ph/9808272].

[6] G.C. Branco, M.N. Rebelo and J.I. Silva-Marcos, *Large neutrino mixing with universal strength of Yukawa couplings*, *Phys. Rev.* **D 62** (2000) 073004 [hep-ph/9906368]; *Degenerate and quasi degenerate majorana neutrinos*, *Phys. Rev. Lett.* **82** (1999) 683 [hep-ph/9810328].

[7] G. 't Hooft, *naturalness, chiral symmetry and spontaneous chiral symmetry breaking*, lecture given at Cargèse Summer Institute 1979, p. 135.

[8] See as an example M. Tanimoto, T. Watari and T. Yanagida, *Democratic mass matrices from broken* $O(3)_L \times O(3)_R$ *flavor symmetry*, *Phys. Lett.* **B 461** (1999) 345 [hep-ph/9904338].

[9] G.C. Branco and J.I. Silva-Marcos, *The symmetry behind extended flavour democracy and large leptonic mixing*, *Phys. Lett.* **B 526** (2002) 104 [hep-ph/0106125].